VARIABILITY REGIONS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. M. Biernacki gave concrete forms of the variability regions of $z/f(z)$ and $zf'(z)/f(z)$ of close-to-convex functions f for a fixed z with $|z| < 1$ in 1936. The forms are, however, not necessarily convenient to determine the shape of the full variability region of $zf'(z)/f(z)$ over all close-to-convex functions f and all points z with $|z| < 1$. We will propose a couple of other forms of the variability regions and see that the full variability region of $zf'(z)/f(z)$ is indeed the complex plane minus the origin. We also apply them to study the variability regions of $\log[z/f(z)]$ and $\log[zf'(z)/f(z)]$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be its subclasses described by the conditions $f(0) = 1$ and $f(0) = f'(0)-1 = 0$, respectively. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. A function f in $\mathcal{A}$ is called *starlike* (resp. *convex*) if f is univalent and if $f(\mathbb{D})$ is starlike with respect to 0 (resp. convex). It is well known that $f \in \mathcal{A}$ is starlike (resp. convex) precisely when $\operatorname{Re}[zf'(z)/f(z)] > 0$ (resp. $\operatorname{Re}[1 + zf''(z)/f'(z)] > 0$) in $|z| < 1$. The classes of starlike and convex functions in $\mathcal{A}$ will be denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively.

A function $f \in \mathcal{A}$ is called *close-to-convex* if $\operatorname{Re}[e^{i\lambda}f'(z)/g'(z)] > 0$ in $|z| < 1$ for a convex function $g \in \mathcal{K}$ and a real constant λ with $|\lambda| < \pi/2$. The set of close-to-convex functions in $\mathcal{A}$ will be denoted by $\mathcal{C}$. This class was first introduced and shown to be contained in $\mathcal{S}$ by Kaplan [6]. A domain is called *close-to-convex* if it is expressed as the image of $\mathbb{D}$ under the mapping $af + b$ for some $f \in \mathcal{C}$ and constants $a, b \in \mathbb{C}$ with $a \neq 0$. He also gave a geometric characterization in terms of turning of the boundary of the domain. We recommend books [3] and [4] for general reference on these topics.

Prior to the work of Kaplan, Biernacki [2] introduced the notion of linearly accessible domains (in the strong sense). Here, a domain in $\mathbb{C}$ is called linearly accessible if its complement is a union of non-intersecting half-lines. Lewandowski [10], [11] proved that the class of close-to-convex domains is identical with that of linearly accessible domains (see also [1] and [8] for simpler proofs of this fact). Therefore, the work of Biernacki on linearly accessible domains and their mapping functions can now be interpreted as that on close-to-convex domains and functions.

For a non-vanishing function g in $\mathcal{A}_0$, unless otherwise stated, $\log g$ will mean the continuous branch of $\log g$ in $\mathbb{D}$ determined by $\log g(0) = 0$. For instance, $f(z)/z$ can be

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C75.

Key words and phrases. close-to-convex function, variability region, linearly accessible.

The present research was supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340025 and (C) 23540209.

regarded as a non-vanishing function in $\mathcal{A}_0$ for $f \in \mathcal{S}$. Therefore, we can define $\log f(z)/z$ in the above sense. In the present note, we are interested in the following variability regions for a fixed $z \in \mathbb{D}$:

$$U_z = \{\tfrac{z}{f(z)} : f \in \mathcal{C}\}, \quad LU_z = \{\log \tfrac{z}{f(z)} : f \in \mathcal{C}\},$$
$$V_z = \{f'(z) : f \in \mathcal{C}\}, \quad LV_z = \{\log f'(z) : f \in \mathcal{C}\},$$
$$W_z = \{\tfrac{zf'(z)}{f(z)} : f \in \mathcal{C}\}, \quad LW_z = \{\log \tfrac{zf'(z)}{f(z)} : f \in \mathcal{C}\}.$$

We collect basic properties of these sets. Here, $\operatorname{Int} E$ means the set of interior points of a subset E of $\mathbb{C}$.

Lemma 1.1.

(1) *X_z is a compact subset of $\mathbb{C}$ for each $z \in \mathbb{D}$ and $X = U, V, W, LU, LV, LW$.*
(2) *$X_z = \exp(LX_z)$ for each $z \in \mathbb{D}$ and $X = U, V, W$.*
(3) *$X_z = X_r$ for $|z| = r < 1$ and $X = U, V, W, LU, LV, LW$.*
(4) *$X_r \subset \operatorname{Int} X_s$ for $0 \le r < s < 1$ and $X = U, V, W, LU, LV, LW$.*

Proof. It is enough to outline the proof since the reader can reproduce the proof easily. Assertion (1) follows from compactness of the family $\mathcal{C}$, whereas (2) is immediate by definition. The property $X_z \subset X_w$ for $|z| \le |w| < 1$ (thus assertion (3)) is implied by the fact that the function $f_a(z) = f(az)/a$ belongs to $\mathcal{C}$ again for $f \in \mathcal{C}$ and $a \in \mathbb{C}$ with $0 < |a| \le 1$. The assertion $\partial X_r \cap \partial X_s = \emptyset$ for $r < s < 1$ is a consequence of the following observation due to Biernacki [2]: The extremal functions in $\mathcal{C}$ for the relevant functionals consist of Koebe transforms of the Koebe function $z/(1-z)^2$. $\square$

Set $X_{1^-} = \bigcup_{0 \le r < 1} X_r$ for $X = U, V, W, LU, LV, LW$. In the sequel, $\mathbb{D}(a, r)$ will stand for the open disk $|z - a| < r$ in $\mathbb{C}$ and $\overline{\mathbb{D}}(a, r)$ will stand for its closure, namely, the closed disk $|z - a| \le r$.

Biernacki [2] described U_z and W_z in his study on linearly accessible domains and their mapping functions. The results can be summarized as in the following.

Lemma 1.2 (Biernacki (1936))**.** *For $0 < r < 1$, the following hold:*

(1) $U_r = \{(1+s)^2/(1+\frac{s+t}{2}) : |s| \le r, |t| \le r\} = \{2u^2/(u+v) : |u-1| \le r, |v-1| \le r\}$.
(2) $W_r = (1-r^2)^{-2} U_r$.
(3) $U_{1^-} = \mathbb{D}(1, 3) \setminus \{0\}$ *and* $LU_{1^-} \subset \{w \in \mathbb{C} : |\operatorname{Im} w| < 3\pi/2\}$.
(4) $LW_{1^-} \subset \{w \in \mathbb{C} : |\operatorname{Im} w| < 3\pi/2\}$.

The above expressions of U_r and W_r are simple but somewhat implicit. For instance, the parametrization of the boundary curve cannot be obtained immediately and the shape of the limit W_{1^-} is not clear (as we will see below, this set is equal to $\mathbb{C} \setminus \{0\}$). Therefore, it would be nice to have more explicit or more convenient expressions of U_r and W_r. We propose two such expressions in the present note.

Theorem 1.3. *For $0 < r < 1$, $U_r = F(\overline{\mathbb{D}}(0, r))$, where*

$$F(z) = \frac{(3+\bar{z})(1+z)^3}{3 + 3z + \bar{z} + z^2}.$$

We will prove the theorem by describing explicitly the envelope of the family of circles $M_s(\partial\mathbb{D}(0,r))$ for $s = re^{i\theta}$, $0 \le \theta < 2\pi$, where M_s is the Möbius transformation $t \mapsto (1+s)^2/(1+(s+t)/2)$. Lewandowski [11, p. 45] used the envelope to prove that the implication $U_r \subset \{w \in \mathbb{C} : \mathrm{Re}\, w \ge 0\}$ (equivalently, $W_r \subset \{w \in \mathbb{C} : \mathrm{Re}\, w \ge 0\}$) is valid precisely when $r \le 4\sqrt{2}-5$. (This implies that the radius of starlikeness of close-to-convex functions is $4\sqrt{2}-5$.) However, he did not need any explicit form of the envelope for that.

We note that $F(e^{i\theta}) = 1 + 3e^{i\theta}$ for $\theta \in \mathbb{R}$, which agrees with Lemma 1.2 (3). But, this does not give enough information to determine the boundary curve of the domain LU_{1^-}. It turns out that LU_{1^-} has relatively a simple description though LU_r does not have. We indeed derive the following result by making use of Theorem 1.3.

Theorem 1.4. *The variability region LU_{1^-} is an unbounded Jordan domain with the boundary curve $\gamma(t)$, $-2\pi < t < 2\pi$, given by*

$$\gamma(t) = \begin{cases} \mathrm{Log}\,(1+3e^{it}) & \text{if } |t| < \pi \\ \mathrm{Log}\,(1-e^{it}) + \dfrac{t}{|t|}\pi i & \text{if } \pi \le |t| < 2\pi. \end{cases}$$

Here and hereafter, $\mathrm{Log}\, w = \log|w| + i\mathrm{Arg}\, w$ denotes the principal branch of $\log w$ with $-\pi < \mathrm{Im}\ \mathrm{Log}\, w = \mathrm{Arg}\, w \le \pi$.

As we will see in the next section, the function $\log F$ is univalent in $\mathbb{D}$. Therefore, the last theorem tells us that $F : \mathbb{D} \to U_{1^-}$ covers the disk $\mathbb{D}(-1,1)$ bivalently whereas it covers $\mathbb{D}(1,3) \setminus (\mathbb{D}(-1,1) \cup \{0\})$ univalently.

The following expression of W_r is not very explicit but useful in some situation.

Theorem 1.5. *For* $0 < r < 1$,

$$W_r = \left\{ \frac{2u}{v(u+v)} : |u-1| \le r, |v-1| \le r \right\}.$$

Indeed, as an application of the last theorem, we can show the following result.

Theorem 1.6. $LW_{1^-} = \{w \in \mathbb{C} : |\mathrm{Im}\, w| < 3\pi/2\}$.

Since $W_{1^-} = \exp(LW_{1^-})$, we obtain the following corollary, which was used in [7].

Corollary 1.7. *The full variability region $\{zf'(z)/f(z) : z \in \mathbb{D}, f \in \mathcal{C}\}$ is equal to $\mathbb{C}\setminus\{0\}$.*

The corollary means that $W_{1^-} = \mathbb{C} \setminus \{0\}$. We note here that this does not seem to follow immediately from Lemma 1.2.

Krzyż [9] gave a concrete description of the variability region LV_r and, in particular, showed that LV_r is convex and contained in the domain $|\mathrm{Im}\, w| < 4\arcsin r$ for each $0 < r < 1$. (See also [4, Chap. 11].) One might expect that LU_r and LW_r would also be convex for each $0 < r < 1$. This is, however, not true unlike LV_r.

Theorem 1.8. *The variability regions LU_r and LW_r are closed Jordan domains for each $0 < r < 1$. Moreover, there exists a number $0 < r_0 < 1$ such that both LU_r and LW_r are not convex for every r with $r_0 < r < 1$.*

We prove the above results in Section 3. Section 2 will be devoted to the study of mapping properties of the function $G = \log F$ that are necessary to show our results.

2. Univalence of the function $G = \log F$

In order to analyze the shape of LU_r or LW_r, we need to investigate mapping properties of the functions F and $G = \log F$, where F is given in Theorem 1.3. Therefore, before showing the main results in Introduction, we see basic properties of the functions F and G. Here, we remark that F can be expressed in the form

$$F(z) = \frac{(1+z)^3}{1 + z\frac{3+z}{3+\bar{z}}}.$$

Therefore, the continuous branch G of $\log F$ with $G(0) = 0$ is represented by

$$G(z) = 3\,\mathrm{Log}\,(1+z) - \mathrm{Log}\,(1 + ze^{2i\phi}), \quad \phi = \mathrm{Arg}\,(3+z). \tag{2.1}$$

The goal in this section is to prove the following:

Theorem 2.1. *The function $G = \log F$ is a homeomorphism of the unit disk $\mathbb{D}$ onto the domain LU_{1^-}.*

For $r \in (0, 1)$ and $x \in (0, \pi)$, we set

$$\Phi_r(x) = \mathrm{Arg}\,(1 + re^{ix}).$$

We will use the following elementary properties of the function Φ_r.

Lemma 2.2. *Let $r \in (0, 1)$. Then*

$$\Phi_r'(x) = \frac{r(r + \cos x)}{1 + 2r\cos x + r^2}.$$

In particular, $\Phi_r(x)$ is increasing in $0 < x < x_r$ and decreasing in $x_r < x < \pi$, where $x_r = \pi - \arccos r$. Furthermore, $\Phi_r'(x)$ is decreasing in $0 < x < \pi$ and therefore Φ_r is concave in $(0, \pi)$.

We also need the following information.

Lemma 2.3. *Let $0 < r < 1$. Then the inequalities $0 < \theta + 2\phi < \pi$ hold for $0 < \theta < \pi$ and $\phi = \mathrm{Arg}\,(3 + re^{i\theta})$.*

Proof. Set $h(\theta) = \theta + 2\phi$. Then

$$\begin{aligned} h'(\theta) &= 1 + \frac{2r(r + 3\cos\theta)}{9 + 6r\cos\theta + r^2} = \frac{3(3 + 4r\cos\theta + r^2)}{9 + 6r\cos\theta + r^2} \\ &\geq \frac{3(3 - 4r + r^2)}{9 + 6r\cos\theta + r^2} = \frac{3(1-r)(3-r)}{9 + 6r\cos\theta + r^2} > 0. \end{aligned}$$

Therefore, $h(\theta)$ is increasing in $0 < \theta < \pi$, which implies that $0 = h(0) < h(\theta) < h(\pi) = \pi$ for $0 < \theta < \pi$. □

As for the function G, we first show its local univalence.

Lemma 2.4. *The function G is orientation-preserving and locally univalent in $\mathbb{D}$.*

Proof. The partial derivatives of G are given by

$$G_z(z) = \frac{6+4z+3\bar{z}+z^2}{(1+z)(3+3z+\bar{z}+z^2)},$$
$$G_{\bar{z}}(z) = \frac{z(3+z)}{(3+\bar{z})(3+3z+\bar{z}+z^2)}.$$

It suffices to show that the Jacobian $J_G = |G_z|^2 - |G_{\bar{z}}|^2$ is positive in $\mathbb{D}$, which is equivalent to the condition $|6+4z+3\bar{z}+z^2| > |z(1+z)|$ in $|z|<1$. If we write $z = re^{i\theta}$, then

$$\begin{aligned} &|6+4z+3\bar{z}+z^2|^2 - |z(1+z)|^2 \\ &= 6(6+14r\cos\theta+4r^2+6r^2\cos 2\theta + r^3\cos\theta + r^3\cos 3\theta) \\ &= 12(1+r\cos\theta)\big((1+r\cos\theta)^2+1-r^2\big) > 0. \end{aligned}$$

Thus we are done. □

Lemma 2.5. *For a fixed* $0<r<1$, *the real part of* $G(re^{i\theta})$ *is a decreasing function in* $0 \le \theta \le \pi$.

Proof. By differentiating both sides of

$$\operatorname{Re} G(re^{i\theta}) = \frac{3}{2}\log(1+2r\cos\theta+r^2) - \frac{1}{2}\log(1+2r\cos(\theta+2\phi)+r^2)$$

with respect to θ, we obtain

$$\frac{\partial \operatorname{Re} G(re^{i\theta})}{\partial\theta} = 3\frac{-r\sin\theta}{1+2r\cos\theta+r^2} - \frac{-r\sin(\theta+2\phi)}{1+2r\cos(\theta+2\phi)+r^2}\left(1+2\frac{\partial\phi}{\partial\theta}\right).$$

Since $\tan\phi = r\sin\theta/(3+r\cos\theta)$, we have the relations

$$\begin{aligned} \frac{\partial\phi}{\partial\theta} &= \frac{r(3\cos\theta+r)}{9+6r\cos\theta+r^2}, \\ \cos 2\phi &= \frac{1-\tan^2\phi}{1+\tan^2\phi} = 1 - \frac{2r^2\sin^2\theta}{9+6r\cos\theta+r^2}, \\ \sin 2\phi &= \frac{2\tan\phi}{1+\tan^2\phi} = \frac{2r(3+r\cos\theta)\sin\theta}{9+6r\cos\theta+r^2}. \end{aligned}$$

We substitute them into the above expression of $\partial(\operatorname{Re} G)/\partial\theta$ and make some simplifications to obtain

$$\begin{aligned} &\frac{\partial \operatorname{Re} G(re^{i\theta})}{\partial\theta} \\ &= -\frac{2(3+4r\cos\theta+r^2\cos 2\theta)(9+12r\cos\theta-4r^2\sin^2\theta-4r^3\cos\theta-r^4)\sin\theta}{9+6r\cos\theta+r^2}. \end{aligned}$$

Note first that

$$3+4r\cos\theta+r^2\cos 2\theta = 1-r^2+2(1+r\cos\theta)^2 > 0.$$

Secondly, put

$$H(r,\theta) = 9+12r\cos\theta-4r^2\sin^2\theta-4r^3\cos\theta-r^4.$$

Then

$$\frac{\partial H(r,\theta)}{\partial\theta} = -4r(3+2r\cos\theta - r^2)\sin\theta.$$

In particular, $H(r,\theta)$ is decreasing in $0<\theta<\pi$ for a fixed $0<r<1$. Therefore,

$$H(r,\theta) \ge H(r,\pi) = (1-r)(3-r)(3-r^2) > 0.$$

We have shown that $\partial(\operatorname{Re} G)/\partial\theta < 0$ for $0<\theta<\pi$. □

We next prove the following:

Lemma 2.6. $\operatorname{Im} G(z) > 0$ *for* $z\in\mathbb{D}$ *with* $\operatorname{Im} z > 0$.

Proof. Fix $r\in(0,1)$. Let $\theta\in(0,\pi)$ and let ϕ be given in (2.1). Note that $0<\phi<\theta$. We need to show that

$$g_r(\theta) = \operatorname{Im} G(re^{i\theta}) = 3\Phi_r(\theta) - \Phi_r(\theta+2\phi) \tag{2.2}$$

is positive. Note that $0<\theta<\theta+2\phi<\pi$ by Lemma 2.3. Lemma 2.2 implies that Φ_r takes its maximum value $\Phi_r(x_r) = \arcsin r$ in $(0,\pi)$. In particular, we have $\Phi_r(\theta+2\phi) \le \arcsin r$. Therefore, $g(\theta) > 0$ for $x_r^- < \theta < x_r^+$, where x_r^- and x_r^+ are the solutions to the equation $3\Phi_r(x) = \arcsin r$ with $0 < x_r^- < x_r < x_r^+ < \pi$.

We next assume that $x_r^+ \le \theta < \pi$. Since Φ_r is decreasing in $x_r^+ < x < \pi$ by Lemma 2.2, we see that $g(\theta) > \Phi_r(\theta) - \Phi_r(\theta+2\phi) > 0$. Finally, we assume that $0<\theta\le x_r^-$. In view of concavity of Φ_r (see Lemma 2.2) together with $\Phi_r(0)=0$, we have $\Phi_r(x_r^-) = \Phi_r(x_r)/3 \le \Phi_r(x_r/3)$. Hence, $x_r^- \le x_r/3$. In particular, we have $\theta+2\phi \le 3\theta \le 3x_r^- \le x_r$. Since Φ_r is increasing and convex in $0<x<x_r$, the inequalities $\Phi_r(\theta+2\phi) < \Phi_r(3\theta) \le 3\Phi_r(\theta)$ follow. Thus we have shown that $g_r(\theta) > 0$ in this case, too. □

The following property will be used for the proof of Theorem 1.4.

Lemma 2.7. *The function* g_r *defined in* (2.2) *satisfies* $g_r'(\theta) > 0$ *for* $0<\theta<x_r = \pi - \arccos r$.

Proof. By definition, we have

$$g_r'(\theta) = 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta+2\phi).$$

Since $1+2\partial\phi/\partial\theta > 0$ (see the proof of Lemma 2.3), Lemma 2.2 implies

$$g_r'(\theta) \ge 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta) = 2\left(1 - \frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta).$$

We note here that

$$1 - \frac{\partial\phi}{\partial\theta} = \frac{3(3+r\cos\theta)}{9+6r\cos\theta+r^2} > 0.$$

Since $\Phi_r'(\theta) > 0$ for $0<\theta<x_r$ by Lemma 2.2, the required assertion follows. □

We are now ready to prove the theorem.

Proof of Theorem 2.1. Since G is orientation preserving and locally univalent by Lemma 2.4, it is enough to show that G is injective on the circle $|z|=r$ for each $r\in(0,1)$. We

note here that G is symmetric in the real axis, in other words, $G(\bar{z}) = \overline{G(z)}$ for $z \in \mathbb{D}$. By Lemmas 2.5 and 2.6, G maps the upper half of the circle $|z| = r$ univalently onto a Jordan arc in the upper half plane. Taking into account the symmetry, we have confirmed that G maps the circle $|z| = r$ univalently onto a Jordan curve which is symmetric in the real axis. Thus the proof is complete. □

3. Proof of main results

In this section, we show the main results presented in Section 1. We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix $0 < r < 1$ and let $s = re^{i\alpha}$. To simplify computations, we consider the set $1/U_r = \{1/w : w \in U_r\}$ instead of U_r. We first observe that the set $\{(1 - \frac{s+t}{2})/(1-s)^2 : |t| \le r\}$ is the closed disk $\overline{\mathbb{D}}(c(\alpha), \rho(\alpha))$, where

$$c(\alpha) = \frac{2-s}{2(1-s)^2} \quad \text{and} \quad \rho(\alpha) = \frac{r}{2|1-s|^2} = \frac{r}{2(1-2r\cos\alpha + r^2)}.$$

Note that

$$c'(\alpha) = \frac{is(3-s)}{2(1-s)^3} \quad \text{and} \quad \rho'(\alpha) = -\frac{r^2 \sin\alpha}{|1-s|^4}.$$

The Biernacki theorem (Lemma 1.2 (1)) together with Lemma 1.1 (4) implies that the boundary curve of $1/U_r$ is the outer envelope of the family of circles $\partial\mathbb{D}(c(\alpha), \rho(\alpha))$, $-\pi < \alpha \le \pi$. We can parametrize the outer envelope as

$$\zeta(\alpha) = c(\alpha) + \rho(\alpha)e^{i\beta(\alpha)}$$

in $-\pi < \alpha \le \pi$. By the symmetry in the real axis and the fact that $\operatorname{Im} c'(0) > 0$, we can take $\beta(\alpha)$ so that $\beta(0) = 0$. Here, $\beta = \beta(\alpha)$ is a real-valued function of α satisfying the condition that $\zeta'(\alpha)$ is tangent to the circle $|w - c(\alpha)| = \rho(\alpha)$ at $\zeta(\alpha)$. In other words, $\zeta'(\alpha) = kie^{i\beta}$ for a real number k. Taking the real part of the relation

$$\zeta'(\alpha)e^{-i\beta} = c'(\alpha)e^{-i\beta} + \rho'(\alpha) + i\beta'(\alpha)\rho(\alpha) = ki,$$

we obtain

$$\operatorname{Re}\,[c'(\alpha)e^{-i\beta}] + \rho'(\alpha) = 0,$$

which implies that

$$\cos(\beta - \arg c'(\alpha)) = \cos(\arg c'(\alpha) - \beta) = -\frac{\rho'(\alpha)}{|c'(\alpha)|} = \frac{2r\sin\alpha}{|1-s||3-s|}.$$

Hence,

$$\begin{aligned} \frac{|c'(\alpha)|}{c'(\alpha)} e^{i\beta} = e^{i(\beta - \arg c'(\alpha))} &= \frac{2r\sin\alpha}{|1-s||3-s|} \pm i\sqrt{1 - \left(\frac{2r\sin\alpha}{|1-s||3-s|}\right)^2} \\ &= \frac{2r\sin\alpha \pm i(3 - 4r\cos\alpha + r^2)}{|1-s||3-s|}. \end{aligned}$$

We recall that $\beta(0) = 0$ and substitute $\alpha = 0$ into this relation in order to eliminate ambiguity of the sign. We then see that the minus sign should be taken there. Hence,

$$e^{i\beta} = \frac{-i(1-\bar{s})(3-s)}{|1-s||3-s|} \cdot \frac{c'(\alpha)}{|c'(\alpha)|} = \frac{s(3-s)(1-\bar{s})^2}{r(3-\bar{s})(1-s)^2}.$$

We now get the form of $\zeta(\alpha)$:

$$\begin{aligned}\zeta(\alpha) &= \frac{2-s}{2(1-s)^2} + \frac{r}{2|1-s|^2} \cdot \frac{s(3-s)(1-\bar{s})^2}{r(3-\bar{s})(1-s)^2} \\ &= \frac{3-3s-\bar{s}+s^2}{(3-\bar{s})(1-s)^3} = \frac{1}{F(-s)}.\end{aligned}$$

Thus the assertion follows. □

Proof of Theorem 1.4. As we saw, the mapping $G = \log F$ is a homeomorphism of $\mathbb{D}$ onto LU_{1^-}. We now observe that G extends continuously to $\overline{\mathbb{D}} \setminus \{-1\}$. Since $G(e^{it}) = \gamma(t)$ for $|t| < \pi$, the boundary of LU_{1^-} contains the arc $\gamma([-\pi, \pi])$.

We next investigate the limit points of $G(z)$ as $z \to -1$. Let $\alpha(\delta) = (a\delta)^{1/3}$ and put $z = (1-\delta)e^{i(\pi-\alpha(\delta))}$ for $0 < \delta < 1$ and a positive constant a. Then

$$\begin{aligned} z &= -(1-\delta)\left(1 - i\alpha(\delta) - \frac{\alpha(\delta)^2}{2} + \frac{i\alpha(\delta)^3}{6} + O(\delta^{4/3})\right) \\ &= -1 + i(a\delta)^{1/3} + \frac{(a\delta)^{2/3}}{2} + \left(1 - \frac{ia}{6}\right)\delta + O(\delta^{4/3})\end{aligned}$$

as $\delta \to 0+$. Therefore,

$$\frac{3+z}{3+\bar{z}} = 1 + i(a\delta)^{1/3} - \frac{(a\delta)^{2/3}}{2} - \frac{2ia\delta}{3} + O(\delta^{4/3})$$

and

$$1 + z\frac{3+z}{3+\bar{z}} = \left(1 + \frac{ia}{2}\right)\delta + O(\delta^{4/3}).$$

Since $(1+z)^3 = -ia\delta + O(\delta^{4/3})$, we have

$$\frac{(1+z)^3}{1 + z\frac{3+z}{3+\bar{z}}} = -\left(1 + \frac{a+2i}{a-2i}\right) + O(\delta^{1/3})$$

as $\delta \to 0+$. Thus

$$\lim_{\delta \to 0+} G((1-\delta)e^{i(\pi-\alpha(\delta))}) = \pi i + \log\left(1 + \frac{a+2i}{a-2i}\right).$$

Since a is an arbitrary positive real number, the boundary of $LU_{1^-} = G(\mathbb{D})$ contains the curve $\gamma(t)$: $\pi < t < 2\pi$. The same is true for $-2\pi < t < -\pi$ by the symmetry of the function G.

The remaining thing is to prove that the boundary of LU_{1^-} in $\mathbb{C}$ contains no other points than the curve $\Gamma = \{\gamma(t) : |t| < 2\pi\}$. We note here that LU_r is convex in the direction of imaginary axis for each $0 < r < 1$ by Lemma 2.5. Therefore, the same is true

for the limit LU_{1^-}. We observe also that Γ encloses an unbounded Jordan domain convex in the direction of imaginary axis.

Suppose that there is a boundary point p_0 of LU_{1^-} with $p_0 \notin \Gamma$. We may assume that $\operatorname{Im} p_0 > 0$. Let p_1 be the point in Γ with $\operatorname{Im} p_1 > 0$ and $\operatorname{Re} p_1 = \operatorname{Re} p_0$. Then the convexity of LU_{1^-} in the direction of imaginary axis implies that the segment $[p_0, p_1]$ is contained in ∂LU_{1^-}. We can choose p_0 so that the segment is maximal. Since the family of smooth Jordan domains LU_r, $0 < r < 1$, exhausts the domain LU_{1^-}, for a small enough $\delta > 0$ there exist three points $z_1^-(\delta), z_0(\delta), z_1^+(\delta)$ on the circle $|z| = 1 - \delta$ with $0 < \operatorname{Arg} z_1^-(\delta) < \operatorname{Arg} z_0(\delta) < \operatorname{Arg} z_1^+(\delta)$ such that $G(z_1^-(\delta)) \to p_1$, $G(z_0(\delta)) \to p_0$, $G(z_1^+(\delta)) \to p_1$ as $\delta \to 0+$. In particular, $\operatorname{Im} G(z_0(\delta)) < \operatorname{Im} G(z_1^\pm(\delta))$ for sufficiently small $\delta > 0$. Therefore, $g_{1-\delta}(\theta) = \operatorname{Im} G((1-\delta)e^{i\theta})$ takes a local minimum at a point θ_0 with $\operatorname{Arg} z_1^-(\delta) < \theta_0 < \operatorname{Arg} z_1^+(\delta)$. In particular, $g'_{1-\delta}(\theta_0) = 0$. Note here that

$$\operatorname{Re} G((1-\delta)e^{i\theta_0}) \to \operatorname{Re} p_0 \quad (\delta \to 0+).$$

We write $\theta_0 = \pi - \beta(\delta)$. Then, by Lemma 2.7, we see that $\theta_0 \geq x_{1-\delta}$, equivalently, $\beta(\delta) \leq \arccos(1-\delta)$. This implies that $\beta(\delta) = O(\delta^{1/2})$ as $\delta \to 0+$. Therefore, $z = (1-\delta)e^{i(\pi - \beta(\delta))} = 1 - i\beta(\delta) + O(\delta)$, $(3+z)/(3+\bar{z}) = 1 + i\beta(\delta) + O(\delta)$ and thus $1 + z(3+z)/(3+\bar{z}) = O(\delta)$ as $\delta \to 0+$. In particular,

$$\operatorname{Re} G((1-\delta)e^{i\theta_0}) \to -\infty \quad (\delta \to 0+),$$

which is a contradiction.

We now conclude that $\partial LU_{1^-} = \Gamma$. □

In order to prove Theorem 1.5, we will make use of a weakened version of Lemma 5.1 in Greiner and Roth [5], which is an outcome of the duality methods developed by Ruscheweyh and Sheil-Small.

For $|a| \leq 1, |b| \leq 1$, define a function $f_{a,b} \in \mathcal{A}_1$ by

$$f_{a,b}(z) = \frac{1 + (a+b)z/2}{(1+bz)^2}.$$

It is easy to see that $f_{a,b}$ belongs to the class $\mathcal{C}$ of close-to-convex functions. The linear space $\mathcal{A}$ is naturally equipped with the topology of uniform convergence on compact subsets in $\mathbb{D}$.

Lemma 3.1 ([5, Lemma 5.1]). *Let λ_1 and λ_2 be continuous linear functionals on $\mathcal{A}$ such that λ_2 does not vanish on $\mathcal{C}$. Then for every $f \in \mathcal{C}$ there exist complex numbers a, b with $|a| \leq 1, |b| \leq 1$ such that*

$$\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})}.$$

Proof of Theorem 1.5. Fix $0 < r < 1$. Let $f \in \mathcal{C}$. We now apply Lemma 3.1 to the choice $\lambda_1(f) = rf'(r)$ and $\lambda_2(f) = f(r)$ to see that

$$\frac{rf'(r)}{f(r)} = \frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})} = \frac{2(1+ar)}{(1+br)(2+(a+b)r)}$$

for some $a, b \in \overline{\mathbb{D}}$. The proof is complete by letting $u = 1 + ar$ and $v = 1 + br$. □

Proof of Theorem 1.6. Let $\Omega = \{(r,s,t) : 0 < s < 2, 0 < rs^2 < 2, -\pi/2 < t < \pi/2\}$. Then $u = rs^2e^{it}\cos^2 t$ and $v = se^{-it}\cos t$ satisfy $|u-1| < 1$ and $|v-1| < 1$, whence the point

$$w(r,s,t) = \log\frac{2u}{v(u+v)} = \log(2r) + 3it - \log(1 + rse^{2it}\cos t)$$

belongs to the region LW_{1^-} for $(r,s,t) \in \Omega$.

For a given point $z_0 = x_0 + iy_0$ with $|y_0| < 3\pi/2$, we now look for $(r,s,t) \in \Omega$ such that $w(r,s,t) = z_0$.

Let $r_0 = e^{x_0}/2$ and take small enough $0 < s_0 < 1$ so that $r_0s_0 < 1/2$. Then $r_0s_0^2 < s_0 < 2$ and $x_0 \pm 3\pi i/2$ are the endpoints of the curve $\alpha(t) = w(r_0, s_0, t)$, $-\pi/2 < t < \pi/2$. We now take a $t_0 \in (-\pi/2, \pi/2)$ such that $\operatorname{Im}\alpha(t_0) = y_0$ and let $x_1 = x_0 - \operatorname{Re}\alpha(t_0)$. Since the function $-\log(1-x)$ is convex, we have the inequality $-\log(1-x) \le 2x\log 2$ for $0 \le x < 1/2$. We now estimate $-x_1$ in the following way:

$$-x_1 = -\log|1 + r_0s_0e^{2it_0}\cos t_0| \le -\log(1 - r_0s_0) \le 2r_0s_0\log 2,$$

which implies

$$r_0s_0^2e^{-x_1} < s_0e^{-x_1} \le s_0e^{2r_0s_0\log 2} < s_0e^{\log 2} = 2s_0 < 2.$$

Therefore $(r_0e^{x_1}, s_0e^{-x_1}, t_0) \in \Omega$ and

$$w(r_0e^{x_1}, s_0e^{-x_1}, t_0) = x_1 + w(r_0, s_0, t_0) = x_0 + iy_0 = z_0$$

as desired. □

Proof of Theorem 1.8.

Since LU_r and LW_r are similar, it suffices to prove the assertion for LU_r. If there is no such an r_0 as in the assertion, then the limiting domain LU_{1^-} must be convex. Note that LU_{1^-} is convex if and only if $\frac{d}{dt}\arg\gamma'(t) \ge 0$, where γ is given in Theorem 1.4. A simple computation gives us

$$\begin{aligned}\frac{d}{dt}\arg\gamma'(t) &= \operatorname{Im}\frac{d}{dt}\log\gamma'(t)\\ &= \operatorname{Re}\frac{1}{1+3e^{it}} = \frac{1+3\cos t}{|1+3e^{it}|^2}\end{aligned}$$

for $|t| < \pi$. This is negative when $\cos t < -1/3$ and thus we get a contradiction. The proof is now complete. □

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